The General Schwarzschild Solution

Hiroaki Igarashi

December 22, 2023

Abstract

As is well known, in deriving the Schwarzschild solution, the coefficients of the metric of the two spheres (i.e. $d\theta^2 + \sin^2\theta d\phi^2$) is often assumed to be r^2. In this paper, this coefficient is taken as an arbitrary function of r, and the "completely general" Schwarzschild solution is derived.

1 Background

The Schwarzschild solution is a solution of Einstein's equations under the assumption that spacetime is spherically symmetric, static, and vacuum. It is generally expressed in the following form (here, the speed of light = 1 and metric signature is $(-, +, +, +)$).

$$ds^2 = -e^\lambda dt^2 + e^\mu dr^2 + e^\nu(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where e^λ, e^μ and e^ν are functions of r only. Assuming $e^\nu = r^2$, and solving Einstein's equations, the metric can be put in the form

$$ds^2 = -\left(1 - \frac{a}{r}\right) dt^2 + \left(1 - \frac{a}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where a is constant.

2 The General Schwarzschild Solution

In this section, we solve the Einstein's equations from line elements of the following form to obtain the completely general Schwarzschild solution.

$$ds^2 = -e^\lambda dw^2 + e^\mu dr^2 + e^\nu(d\theta^2 + \sin^2\theta d\phi^2),$$

where e^λ, e^μ and e^ν are functions of r only and assumed to be not constant.

The connection coefficients are as follows except for 0.

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\lambda', \Gamma^1_{00} = -\frac{1}{2}e^{\lambda-\mu}\lambda', \Gamma^1_{11} = \frac{1}{2}\mu',$$
$$\Gamma^1_{22} = -\frac{1}{2}\nu' e^{\nu-\mu}, \Gamma^1_{33} = -\frac{1}{2}\nu' e^{\nu-\mu}\sin^2\theta, \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\nu',$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta, \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}\nu', \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta},$$

where the primes denote differentiation with respect to r.

The Ricci tensors can be obtained by the following formula.

$$R_{\mu\nu} = \partial_\mu \Gamma^\alpha_{\alpha\nu} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\tau}\Gamma^\tau_{\nu\alpha} - \Gamma^\alpha_{\alpha\tau}\Gamma^\tau_{\mu\nu}$$

.

From $R_{00} = R_{11} = R_{22} = R_{33} = 0$, we have

$$-\lambda'\mu' + 2\lambda'' + \lambda'^2 + 2\lambda'\nu' = 0, \tag{2}$$
$$-\lambda'\mu' + 2\lambda'' + \lambda'^2 + 4\nu'' + 2\nu'^2 - 2\mu'\nu' = 0, \tag{3}$$
$$2\nu'' + 2\nu'^2 + \lambda'\nu' - \mu'\nu' = 4e^{\mu-\nu}. \tag{4}$$

The other R_{ij} are identically 0.

From equations (2) and (3) and $\nu' \neq 0$, we obtain

$$\mu' - \nu' = -\lambda' + \frac{2\nu''}{\nu'}, \tag{5}$$

and

$$\mu - \nu = -\lambda + 2\log|\nu'| + C, \tag{6}$$

where C is the constant of integration.

Also, by equation (4),

$$2\nu'' - \nu'(\mu' - \nu') + \nu'^2 + \lambda'\nu' = 4e^{\mu-\nu}. \tag{7}$$

Substituting equations (5) and (6) for equation (7), results in

$$(C_1 e^{-\lambda} - 1)\nu'^2 = 2\lambda'\nu', \tag{8}$$

where $C_1 = 4e^C$.

$C_1 e^{-\lambda} - 1$ is not identically 0 since λ is not constant[1] and $C_1 > 0$. Hence, from equation (8), we obtain

$$\nu' = \frac{2\lambda'}{C_1 e^{-\lambda} - 1}. \tag{9}$$

[1]If λ is constant, we have $\lambda = 0$ and $\mu = \nu + 2\log|\nu'| - \log 4$ with ν as any function of r only.

By equations (9) and (5), we have

$$\mu' = \frac{(2\lambda'' + \lambda'^2)C_1 e^{-\lambda} - 2\lambda'' + 3\lambda'^2}{\lambda'(C_1 e^{-\lambda} - 1)}. \tag{10}$$

Let $f(r) = e^{\lambda}$ (> 0). Then,

$$\mu' = \frac{C_1(2f''(r)f(r) - f'(r)^2) + f(r)(-2f''(r)f(r) + 5f'(r)^2)}{f(r)f'(r)(C_1 - f(r))}, \tag{11}$$

$$\nu' = \frac{2f'(r)}{C_1 - f(r)}. \tag{12}$$

Therefore,

$$\mu = \log\frac{f'(r)^2}{f(r)(C_1 - f(r))^4} + C_2, \tag{13}$$

$$\nu = -2\log|C_1 - f(r)| + C_3, \tag{14}$$

where C_2 and C_3 are the constant of integration.

Consequently,

$$e^{\lambda} = f(r), \tag{15}$$

$$e^{\mu} = \frac{f'(r)^2}{f(r)(C_1 - f(r))^4}C_2, \tag{16}$$

$$e^{\nu} = \frac{1}{(C_1 - f(r))^2}C_3 \tag{17}$$

(here, we denote $C_2 = e^{C_2}$ and $C_3 = e^{C_3}$).

3 Conclusion

The general solution of Einstein's equations under the assumption that any metric is not constant and spacetime is spherically symmetric, static, and vacuum is as follows.

$$ds^2 = -f(r)dt^2 + \frac{f'(r)^2}{f(r)(C_1 - f(r))^4}C_2 dr^2 + \frac{1}{(C_1 - f(r))^2}C_3(d\theta^2 + \sin^2\theta d\phi^2),$$

where C_1, C_2 and C_3 are constant and $f(r)$ is any C^2 function of r only and $f(r) > 0$.

Example 1. If $f(r) = 1 - \frac{a}{r}$, $C_1 = 1$, $C_2 = C_3 = a^2$, we obtain the solution in Schwarzschild coordinates ($r > a > 0$).

$$ds^2 = -\left(1 - \frac{a}{r}\right)dt^2 + \left(1 - \frac{a}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Example 2. If $f(r) = \left(\frac{r-a}{r+a}\right)^2$, $C_1 = 1$, $C_2 = C_3 = 16a^2$, we obtain the solution in isotropic coordinates ($r > 0$, $a > 0$, $r \neq a$).

$$ds^2 = -\left(\frac{r-a}{r+a}\right)^2 dt^2 + \left(1 + \frac{a}{r}\right)^4 dr^2 + \left(1 + \frac{a}{r}\right)^4 r^2(d\theta^2 + \sin^2\theta d\phi^2).$$